Local Regularization for the Nonlinear Inverse Autoconvolution Problem

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In this paper we develop a local regularization theory for the inverse autoconvolution problem of finding $x \in L_2(0,T)$ solving

$$G(x) = f, \tag{1}$$

where G is the nonlinear Volterra operator given by

$$G(x)(t) = \int_0^t x(t-s)\,x(s)\,ds, \quad \text{a.e. t} \in (0,\text{T}), \tag{2}$$

and where $f \in Range(G) \subseteq L_2(0,T)$.

The local regularization method is suggested by the idea of momentarily holding x constant on a small local interval $[t, t+R]$. Here, the length of this local interval R serves as the regularization parameter. The regularization equation is then given by

$$\alpha_R(x)x + F_R(x) = f_R^\delta, \tag{3}$$

where for a.e. $t \in (0,T)$,

$$\begin{aligned}
\alpha_R(x) &\equiv 2\int_0^R \int_0^\rho x(s)\,ds\,d\eta(\rho), && (4)\\
F_R(x)(t) &\equiv \int_0^R \int_\rho^t x(t+\rho-s)\,x(s)\,ds\,d\eta(\rho), && (5)\\
f_R^\delta(t) &\equiv \int_0^R f^\delta(t+\rho)\,d\eta(\rho), && (6)
\end{aligned}$$

with the positive Borel measure η selected depending on the parameter R.

Unlike classical regularization techniques such as Tikhonov regularization, this theory provides regularization methods that preserve the causal nature of the autoconvolution problem, allowing for fast sequential numerical solution. We prove the convergence of the regularized solutions to the true solution as the noise level in the data shrinks to zero and supply a convergence rate. We propose several regularization methods and provide a theoretical basis for their convergence; of note is that this class of methods does not require an initial guess of the unknown solution. Our numerical results confirm effectiveness of the methods, suggesting superiority of our methods over the existing ones in the literature, especially in recovering sharp features in the solution.